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Von Neumann's Informal Hidden-Variable Argument

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Von Neumann was convinced that the randomness observed in quantum mechanical systems is inherent in them and not due to an ignorance of additional random variables unspecified in the quantum formalism. His formal "proof" of the nonexistence of these hidden variables,[1] however, relied on overly restrictive assumptions concerning their nature and thus must be considered unacceptable.[2]

For historical perspective, Wigner[3] has recently described in this Journal the informal argument which motivated Von Neumann to his conviction. Also presented is Schrödinger's objection to his reasoning, but in a manner which suggests that the objection is untenable. It is the purpose of the present note to show that Schrödinger's objection is valid and that Von Neumann's motivating argument is also unacceptable.

Von Neumann's argument[4] concerns successive measurements of different spin components of a spin-$\frac{1}{2}$ particle, with the assumption that the result is determined by a hidden variable (or set of variables).

It may be stated briefly as follows:

1. A single measurement yielding a given sign will restrict the range of values which the hidden variable(s) had before the measurement.
2. The restriction will be present after the conclusion of the measurement, otherwise successive measurements of the same component would not yield the same result.
3. A subsequent measurement operation of a different spin component will further restrict this range.
4. A sufficiently large number of these operations will allow the production of a state for which the spin components have a definite sign in all directions.
5. The resultant state will violate the predictions of the quantum theory, and no such violations have been observed.

Schrödinger objected to Von Neumann's reasoning. He suggested that a later measurement, while further restricting the range of the hidden variable(s) may restore a range blocked by an earlier measurement. He thus felt that such a restoration allowed the predictions of quantum mechanics for a spin-$\frac{1}{2}$ particle to be achieved by a hidden-variable theory.

Von Neumann and Wigner counter with two assertions.[4] First they claim that such a restoration "...presupposed the existence of hidden variables in the apparatus used for the measurement." Second they assert that the existence of these hidden variables still allows the generation of a state with well-defined spin components in all directions. Thus they believe that they have refuted Schrödinger's objection.

In this note both of these assertions are demonstrated to be false. A trivial counter example is provided which accomplishes Schrödinger's restoration without requiring the existence of hidden variables in the apparatus. The model is capable of duplicating the predictions of quantum mechanics for an arbitrary series of spin component measurements of a spin-$\frac{1}{2}$ particle. Obviously, since the existence of hidden variables in the apparatus is unnecessary for the measurement operation, the second assertion is likewise untrue, as the apparatus may choose simply to ignore their existence.

Consider an ensemble of spin-$\frac{1}{2}$ particles which are polarized along the direction $\mathbf{p}$. The polarization direction is characteristic of and carried by every member of this ensemble. Assume that each member of the ensemble also has a hidden variable which is the unit vector $\boldsymbol{\lambda}$, and that $\boldsymbol{\lambda}$ has initially a uniform probability distribution over the hemisphere $\boldsymbol{\lambda}\cdot\mathbf{p}\geq 0$.

Next consider an apparatus which measures the spin component along the unit vector $\mathbf{a}$. The action of the apparatus is twofold. First it must be sensitive to the information conveyed to it by the particle (in this case $\boldsymbol{\lambda}$ and $\mathbf{p}$), and from this information determine a binary result $A(\mathbf{a}, \mathbf{p}, \boldsymbol{\lambda})=\pm 1$. Second it must prepare the state for future measurements, without the use of any additional random variables intrinsic to the apparatus.

Construction of a model for the first part of this operation is straightforward, and has already been done by Bell.[5,6] Define

$$\theta \equiv \cos^{-1}(\mathbf{a}\cdot\mathbf{p}),$$

and construct a new vector $\mathbf{a}'$ in the plane of $\mathbf{a}$ and $\mathbf{p}$, defined so that

$$\theta' \equiv \cos^{-1}(\mathbf{a}'\cdot\mathbf{p}) = \tfrac{1}{2}\pi(1-\cos\theta)$$

as shown in Fig. 1.

Now specify the result of the measurement to be

$$A(\mathbf{a}, \mathbf{p}, \boldsymbol{\lambda}) = \operatorname{sgn}(\boldsymbol{\lambda}\cdot\mathbf{a}').$$

Averaging over $\boldsymbol{\lambda}$ yields the expectation value

$$\langle m_p = +\tfrac{1}{2} \mid \boldsymbol{\sigma}\cdot\mathbf{a} \mid m_p = +\tfrac{1}{2}\rangle = 1-(2\theta'/\pi) = \cos\theta$$

in agreement with the predictions of quantum mechanics.

The preparation of the new state for a subsequent measurement must now be done. We shall consider

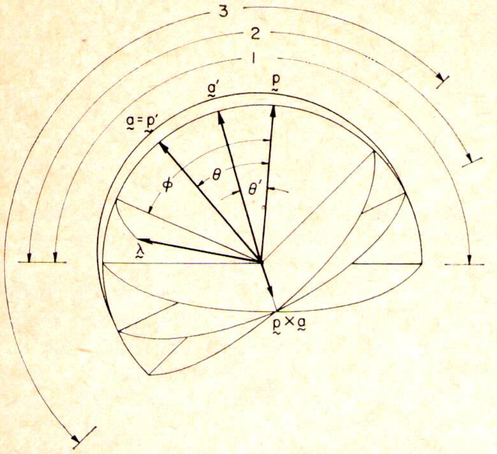

FIG. 1. Hidden-variable phase space: (1) Initial domain of $\boldsymbol{\lambda}$; (2) portion of initial domain for which $A=+1$; (3) final domain of $\boldsymbol{\lambda}$.

the case of a measurement apparatus that passes only particles for which the result of the measurement is $A=+1$. All of these have $\boldsymbol{\lambda}$ within the intersection of the two hemispheres $\boldsymbol{\lambda}\cdot\mathbf{p}\geq 0$ and $\boldsymbol{\lambda}\cdot\mathbf{a}'\geq 0$. Define ϕ to be the azimuthal angle of $\boldsymbol{\lambda}$ referenced from $\mathbf{p}$ about the $\mathbf{p}\times\mathbf{a}$ axis (see Fig. 1), and prescribe that the measurement apparatus rotate $\boldsymbol{\lambda}$ about the $\mathbf{p}\times\mathbf{a}$ axis, keeping the polar angle fixed, to a new azimuthal angle given by

$$\phi'=\frac{\phi}{1-\theta'/\pi}+\theta-\frac{\theta'}{2(1-\theta'/\pi)}\,.$$

By doing so the initial space is mapped on to the hemisphere $\boldsymbol{\lambda}\cdot\mathbf{a}\geq 0$. Finally prescribe that the apparatus define $\mathbf{p}'=\mathbf{a}$ as the new polarization direction after the measurement operation.

The above deterministic procedure assures that the distribution of $\boldsymbol{\lambda}$ after the measurement will be uniform over the hemisphere $\boldsymbol{\lambda}\cdot\mathbf{p}'\geq 0$. Thus the new hidden-variable distribution will be identical to that before the measurement, only rotated to the new orientation in the direction of $\mathbf{p}'=\mathbf{a}$.

A second measurement following a similar set of prescriptions for the direction $\mathbf{b}$ will then yield the expectation value

$$\langle m_{\mathrm{p}'}=+\tfrac{1}{2}\,|\,\boldsymbol{\sigma}\cdot\mathbf{b}\,|\,m_{\mathrm{p}'}=+\tfrac{1}{2}\rangle=\mathbf{b}\cdot\mathbf{p}',$$

again in agreement with the predictions of quantum mechanics. Nowhere in our example is there any need of external (apparatus) hidden variables.

The above trivial example serves to demonstrate that a hidden-variable theory is capable of yielding the predictions of quantum mechanics for an arbitrary series of measurements of different spin components of a spin-$\frac{1}{2}$ particle. Thus Von Neumann's informal argument is also invalid, as well as his formal one.

A brief comment is warranted concerning the relation between Von Neumann's informal argument and Wigner's discussion of Bell's theorem. Consider nine experiments consisting of two successive spin component measurements of a single spin-$\frac{1}{2}$ particle, each measurement made in one of the three directions $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, and $\boldsymbol{\omega}_3$. A given experiment will have four possible outcomes, each corresponding to a different range of the hidden-variable initial values. The nine experiments will define 4^9 essentially different regions.

Wigner[3] has observed that Bell's locality postulate, requiring the result at either measurement to be independent of the orientation of the other apparatus, reduces the number of essentially different regions to 2^6. Von Neumann's informal argument requires that each successive partitioning of the hidden-variable domain (by each successive measurement) commute with other partitionings. This assumption then reduces the number of essentially different regions to 2^3. The predictions of the quantum theory will be violated in either case. Bell's locality postulate achieves a reduction in the number of regions through an eminently reasonable assumption. Von Neumann's assumption on the other hand, is seen to be unreasonable in light of Schrödinger's objection and the above counter example. Thus Wigner has successfully recast Bell's theorem in a form parallel to Von Neumann's informal argument, but in a way which remedies the faults of the latter. Unfortunately the important experimental predictions of Bell's theorem are as yet untested.[7] It is hoped that this note will put these arguments into their proper historical perspective.

The author wishes to acknowledge helpful comments by Dr. A. Shimony.

[1] J. Von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932) [English translation: *Mathematical Foundations of Quantum Mechanics* (Princeton U. P., Princeton, N. J., 1955)].

[2] J. S. Bell, Rev. Mod. Phys. **38**, 447 (1966); D. Bohm and J. Bub, *ibid.* **38**, 453 (1966).

[3] E. P. Wigner, Amer. J. Phys. **38**, 1005 (1970).

[4] The form of Von Neumann's informal argument referred to in this article is that as related by Wigner in Ref. 3.

[5] J. S. Bell, Physics **1**, 195 (1965).

[6] Indeed models have been published earlier which duplicate the predictions of quantum mechanics for a succession of spin component measurements. Notable of these efforts was that of D. Bohm and J. Bub, Ref. 2. Their model was not used in the present discussion for two reasons. First, it includes a dynamical description of the wave function collapse. The associated finite collapse time may for short times yield predictions at variance with quantum theory. Second, evolution of the new distribution for hidden variables is left unspecified, and presumably occurs through external random interactions.

[7] J. F. Clauser, Bull. Amer. Phys. Soc. **14**, 578 (1969); J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Letters **23**, 880 (1969).